2.1 The Function Defined: Linear and Quadratic Functions

Definition 1: Suppose A and B are two non-empty sets. A *Function* assigns to every element of the set A exactly one element of the set B.

As with every new definition or concept introduced in mathematics, it is best to re-read the information and accept it as your new understanding. Trying to make it fit your logic is ineffective. You move your head to fit the logic, not the logic to fit your head. You can always change your mind. You cannot change logic, facts, or the expression $2 + 2 = 4$. More will become clear as you continue with this perspective throughout your practice.

Definition 2: The *Domain* of a function is the set of all possible inputs for that function.

Definition 3: The *Range* of a function is the set of all possible outputs for that function.

In Definition 1, set A is the Domain of the function. Set B is the Range of the function.

Note that the range depends entirely on the domain of the function. This is why we refer to the domain as the set of all independent variables, and the range as the set of all dependent variables.

There is a visual way to interpret Definition 1. Essentially it is saying is that for each x_1 in the domain of some function $f(x)$ there is exactly one single $f(x_1)$ in the range of the function. The visual interpretation of this concept is called the *Vertical Line Test* or *V.L.T.*.

Vertical Line Test: A curve $y = f(x)$ in the Cartesian plane is a function if and only if any vertical line that passes through that curve intersects the curve in at most one point.

We will return to this concept after we have discussed the function in greater detail.

Before jumping into constant, linear, and quadratic functions, let us discuss the most fundamental part of a function: the Domain.

Many students have difficulty determining the domain of a function. Often in mathematics it seems that there are an infinite number of answers, and sometimes this is true. This may be very daunting and extremely overwhelming, often to the point that we feel paralyzed. This tells us that it is the wrong approach. We can always categorize problems and from that create finite lists of solutions. Instead of concentrating on every function ever known and the corresponding domains, let us just consider those that give us trouble. Fortunately, given the scope of our study here, there are only three types of functions to consider.

(1) $f(x) = \frac{1}{x}$**, here it must be that** $x \neq 0$**.**

Under no circumstance may we divide by zero. The domain is written in set notation as $D_f = \{x \mid x \neq 0\}$**.**

(2) $g(x) = \sqrt[n]{x}$**, where when** n **is an even number,** $x \geq 0$**.**

We may never take the even root of a negative number. Often we hear the square root of -1 **referred to as the *imaginary number* or** $\sqrt{-1} = i$**. It is true that the square root of a negative number is imaginary and it does not exist in the real numbers. The domain is written in set notation as** $D_g = \{x \mid x \geq 0\}$**.**

(3) $h(x) = log_a(x)$**, here it must be that** $x > 0$**.**

That is to say that you can never take the logarithm of zero or a negative number. The Domain is written in Set Notation as $D_h = \{x \mid x > 0\}$

If the function does not fall under one of these three categories, then we may assume that it has a domain of all real numbers. This is written in set notation as $\{x \mid x \in \mathbb{R}\}$ **or in interval notation as** $(-\infty, \infty)$**.**

Example 1: Find the domain for each of the following functions.

(a) $f(x) = \dfrac{7+x}{x+2}$

This type of function is known as a rational function and we will be discussing it in much greater detail later. Notice that $f(x)$ has an expression of x in the denominator. This will allow the denominator to equal zero when $x = -2$. It is necessary that we take this value of x out of play. In other words, $x = -2$ is not in the domain of the function $f(x)$. We write this in set notation as $D_f = \{x \mid x \neq -2\}$.

(b) $g(x) = \sqrt{x-5}$

Here we have the expression $x - 5$ under an even root. In particular, the root of 2 , also known as the square-root. According to our rules regarding domains we must have $x - 5 \geq 0$. This is what we want, as opposed to Example 1 (a), where we were looking for what to not include in the domain. We solve the inequality $x - 5 \geq 0$ to get $x \geq 5$, which are the values we want to include in our domain. Written in set notation, the domain is $D_g = \{x \mid x \geq 5\}$.

(c) $h(x) = x^2 + 2x + 1$

Since this function has no denominator that can equal zero, it is not an even root, and it is certainly not a logarithm, then we can conclude that the domain is the set of all real numbers. That is $D_h = \{x \mid x \in \mathbb{R}\}$. As you may recall, $h(x) = x^2 + 2x + 1$ is a quadratic function, which is a polynomial function of degree 2 (the highest power of x is 2), and polynomial functions exist everywhere. This means that they will always have the domain of all real numbers.

Finding the range of different functions will be covered in last section of Chapter 2 when we discuss inverse functions.

The most basic functions are constant functions and belong to the family of functions known as polynomials. Constant functions equal a number, like $y = 2$. These are simple functions that do not change. In fact, they are simply horizontal lines. Constants have no rate of change. That is why they are referred to as constants.

Linear functions are the next tier of functions that belong to the family of functions known as polynomials. We often see these represented in slope intercept form. That is, $y = m \cdot x + b$, where m is the slope and b is the ***y-intercept***, which occurs when $x = 0$, or $f(0) = b$.

Example 2: Given the function $f(x) = 2 \cdot x + 1$, find $f(3)$.

While this seems to be possibly the easiest problem we have encountered yet, do not take it for granted. Remember that mathematics is a discipline, and so it requires nurturing healthy habits at the most basic levels in order to become proficient at more difficult problems. We must now get into the habit of using parentheses when evaluating functions. This will be crucial when it comes to understanding the basic concepts of calculus. We develop these habits now, and much will make sense in the future. One reason mathematics is so disliked is because we have been doing it wrong for so long that the wrong way has become accepted and mythic terms like "non-math people" have emerged.

When we encounter a function, we will rewrite the function by replacing the x's with parentheses. $f(x) = 2\cdot x + 1$ will be re-written as $f(\ ) = 2\cdot(\ ) + 1$. This small change to how we approach functions will mean a world of difference when we begin calculus. It will actually change the way our brain interprets the function.

Now to find $f(3)$, you need only fill in the blanks of the function. $f(3) = 2\cdot(3) + 1 = 7$. The element 3 is from the domain of $f(x)$ and yields the value 7 from the range of $f(x)$.

Remember that these subtleties are extremely valuable in our practice. Perfect practice makes perfect.

Example 3: Given the function $f(x) = x^2 + 4\cdot x + 4$, find $f(2)$.

We approach this the same way by replacing all of the x's with blank spaces in parentheses.

Then $f(\ ) = (\ )^2 + 4\cdot(\ ) + 4 \Rightarrow f(2) = (2)^2 + 4\cdot(2) + 4 \Rightarrow 4 + 8 + 4 \Rightarrow 16$

Get into this habit every time you evaluate a function. Have faith in these methods, no matter how silly they may seem. Remember, you are in the process of rewiring your brain for calculus.

The Average Rate of Change (ARC) of a function, also known as slope, is the vertical change of a linear function over the horizontal change. Often written as $m = \frac{\triangle y}{\triangle x}$, where $\triangle$ is the capital Greek letter delta, which in mathematical terms means "change."

Given two independent variables $x_1 \neq x_2$, we find the slope as follows.

$m = \dfrac{\triangle y}{\triangle x} = \dfrac{f(x_1) - f(x_2)}{x_1 - x_2} = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$ **, notice that the respective order of the numerator and denominator must remain consistent. That is, if you start with $f(x_1)$ on top, then you must start with x_1 on the bottom, and vice versa.**

Example 4: Given $g(x) = x^2 - 4$, calculate the average rate of change (slope) from $x = 1$ to $x = 3$. Then find the equation of the secant line containing the points $(1, g(1))$ and $(3, g(3))$. Write your final equation in the form $y = mx + b$. Be sure to show all work and do not skip steps.

First calculate $g(1)$ and $g(3)$. Remember not to deviate from the method of replacing the x's with blank parentheses $g(\,) = (\,)^2 - 4$.

Then $g(1) = (1)^2 - 4$ and $g(1) = -3.$

Likewise $g(3) = (3)^2 - 4$ and $g(3) = 5$.

This gives us two points: $(1, g(1)) = (1, -3)$ and $(3, g(3)) = (3,\ 5)$.

Now calculate $m = \dfrac{g(1) - g(3)}{(1) - (3)} = \dfrac{(-3) - (5)}{(1) - (3)} = \dfrac{-8}{-4} = 2$.

Recall that $y - y_1 = m\cdot (x - x_1)$, and we just pick a set of points, plug them in, and simplify.

$y - 5 = 2\cdot (x - 3) \Rightarrow y = 2\cdot x - 6 + 5 \Rightarrow y = -2\cdot x - 1$

This is the method we always use to find the equation of a line. First make sure you have your points, then calculate the slope. After that, plug the slope and one set of points (does not matter which) into the point-slope form. That is $y - y_1 = m\cdot (x - x_1)$. This is the form to always start with. Always end with slope intercept form $y = m\cdot x + b$. Never begin with slope intercept form. This is bad practice.

This is a good problem to rehearse at least five times as you will see it often in calculus.

The next important set of functions are called quadratic functions. These are a type of polynomial and as we have shown polynomials exist everywhere. This means that their domain is the set of all real numbers. The standard form of a quadratic function is $f(x) = a{\cdot}\, x^2 + b{\cdot}\, x + c$, where a, b, and c are real numbers and $a \neq 0$.

Quadratics are best known for their parabolic shape, as we discussed in the first chapter. They also express a minimum when $a > 0$ where the parabola is concave up, and a maximum when $a < 0$ where the parabola is concave down. Parabolas have a number of applications. In calculus, we will be very concerned with the extremum of curves and functions, which is just an all-encompassing way of speaking about their maximum and minimum values. There will be fancy ways to calculate these for different functions, but for now let us just concentrate on the maximum and minimum values of the quadratic function. This maximum or minimum point on a parabola (quadratic) is known as the vertex, and it is relatively easy to find. A concave up parabola occurs when $a > 0$, and concave down when $a < 0$. Some like to think of a happy face when positive and a sad face when negative. A minimum and a maximum are found in each case, respectively.

We can find the vertex of the parabola as a point in the Cartesian plane by completing the square and putting the function in vertex form. This is similar to the completing the square method we introduced in Chapter 1.2, but this time the function is not identically zero. This makes things different, but in the end the process is much easier. Just as we did in Chapter 1.2, if we practice this method beginning with the function in standard form with the generalized terms $f(x) = a{\cdot}\, x^2 + b{\cdot}\, x + c$, we can do this for any values of a, b, and c, with $a \neq 0$. Moreover, we will be able to graph quadratic functions using transformations in the next section when we have it in this form, known as the vertex form of the quadratic.

Example 5: Given the quadratic equation in standard form $f(x) = a{\cdot}\, x^2 + b{\cdot}\, x + c$, where a, b, and c are any real numbers and $a \neq 0$, write the function in vertex form: $a{\cdot}\,(x - h)^2 + k$, where $(h,\, k)$ is the vertex.

Remember when practicing these new methods, you want to have a mantra to speak out loud with each step. Otherwise, the movements are just empty and the work is less productive.

$f(x) = a{\cdot}\, x^2 + b{\cdot}\, x + c$

$= a\cdot(x^2 + \frac{b}{a}\cdot x) + c$ **"Factor a from the first two terms."**

$= a\cdot(x^2 + \frac{b}{a}\cdot x + \frac{b^2}{4\cdot a^2}) + c - a\cdot\frac{b^2}{4\cdot a^2}$ **"Take the coefficient of x , divide it by 2, and then square it. Add this to the inside and then make up for it on the outside by multiplying the resulting term by a and subtracting it."**

$= a\cdot(x + \frac{b}{2\cdot a})^2 + c - \frac{b^2}{4\cdot a}$ **"Write the terms in parentheses as a perfect square".**

$= a\cdot(x - h)^2 + k$

The above is the vertex form of the quadratic.

Notice that $h = \frac{-b}{2\cdot a}$. Less important is that $k = c - \frac{b^2}{4\cdot a}$, since the easiest way to calculate k is by finding h and the noting that $k = f(h)$. Thus $(h,\ k) = (h,\ f(h))$ is the vertex of the quadratic function.