2.2 Graphing Functions Using Transformations

In Chapter 1.1 we discussed Seven Basic Graphs. In this section, we will use a variety of techniques to graph any variation of the Seven Basic Graphs needed for this course. This will seem difficult at first, as do many things in mathematics. However, we will employ easily understood methods to make this seemingly complex idea quite simple.

Horizontal Shifting:

Horizontal Shift Left: Suppose that $c > 0$ **is a real number and that** $f(x)$ **is a function. The new function** $g(x) = f(x + c)$ **is just** $f(x)$ **shifted left** c **units.**

Horizontal Shift Right: Suppose that $c > 0$ **is a real number and that** $f(x)$ **is a function. The new function** $h(x) = f(x - c)$ **is just** $f(x)$ **shifted right** c **units.**

This is a very simple concept. In Chapter 1.1 we declared at least two known points on each graph. To perform a Horizontal Shift left or right, we need only add or subtract c **to the** x **component, respectively. These shifts must also include the Vertical Asymptote when applicable.**

We begin with a discussion of the Horizontal Shift because when we perform a series of Transformations, we want to work from the inside out. This is a good practice that will greatly reduce the occurrence of silly mistakes and greatly effect our overall knowledge of the subject. It is the little things that when practiced over and over yield great results. Throughout this text we have discussed the need for a minimalist approach to become more efficient at Algebra and simple arithmetic. If we remember that the Horizontal Shift Left is an addition of a positive constant inside the function, then it must following that subtracting the same constant on the inside of the function will have the opposite effect. Thus resulting in a Horizontal Shift Right. Then we only need to remember one concept instead of two. The second comes from a good knowledge of the first. It is a common mistake to reverse the shifting here, since addition moves right on the number line. But this is not the number line. This is the inside of a function, and this is where we start when transforming a function.

Vertical Shifting:

Vertical Shift Up: Suppose that$c > 0$ is a real number and that $f(x)$ is a function. The new function $g(x) = f(x) + c$ is just $f(x)$ shifted up c units.

Vertical Shift Down: Suppose that$c > 0$ is a real number and that $f(x)$ is a function. The new function $h(x) = f(x) - c$ is just $f(x)$ shifted down c units.

Again, this is a very simple concept. In Chapter 1.1 we declared at least two known points on each graph. To perform a Vertical Shift up or down, we need only add or subtract c to the y component, respectively. These shifts must also include Horizontal Asymptotes when applicable.

Throughout this text we repeat the discussion for the need of a minimalist approach to become more efficient at Algebra and simple arithmetic. If we just remember that the Vertical Shift Up is an addition of a positive constant outside of the function, then it must following that subtracting the same constant on the outside of the function will have the opposite effect. Thus resulting in a Vertical Shift Down. We only need to remember one concept instead of two. The second comes from a good knowledge of the first.

Example 1: Graph the function $h(x) = \frac{1}{x-1} + 2$

We want to approach this example methodically, first beginning with the most basic function from which $h(x)$ is composed. We give it another name to avoid confusion.

Let $f(x) = \frac{1}{x}$.

We will graph this function just as we did in Chapter 1.1 making sure to plot two very important points $(-1, -1)$ and $(1, 1)$, as well as the Vertical Asymptote $x = 0$ and the Horizontal Asymptote $y = 0$.

Remember that Vertical and Horizontal Asymptotes must undergo Horizontal and Vertical Shifts, respectively. You may want to read that sentence again to make sure you understand what it means. It is not a mistake that Vertical Asymptote shifts left and right, and that Horizontal Asymptotes shift up and down. That is one of the qualities that make this such a great first example.

Working from the inside out, we subtract1 **from the inside of the original function and shift the original graph right one unit concentrating only on the** ***x-coordinate.***

That is, $g(x) = f(x-1) = \frac{1}{x-1}$**.**

So $f(x) \longmapsto g(x)$**, and we have that** $(-1,-1) \longmapsto (0,-1)$ **and** $(1,1) \longmapsto (2,1)$ **and the Vertical Asymptote** $x = 0 \longmapsto x = 1$**.**

The Horizontal Asymptote is left unchanged at $y = 0$**. If we work from the inside of the function to the outside, we can focus on one thing at a time.**

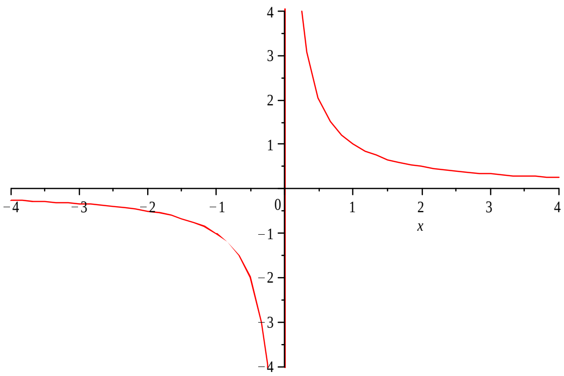
The next step is to implement the Vertical Shift. Here concentrate only on ***y-coordinate*** **and the Horizontal Asymptote.**

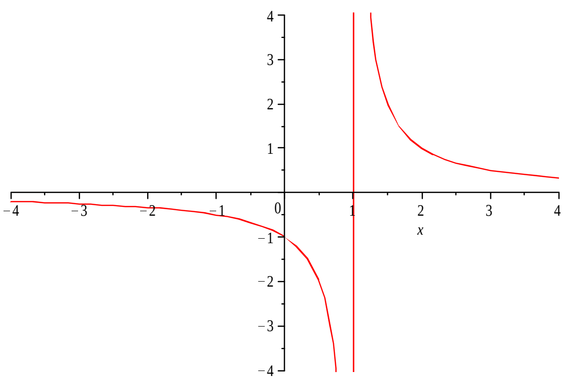
Let $h(x) = g(x) + 2 = f(x-1) + 2 = \frac{1}{x-1} + 2$**.**

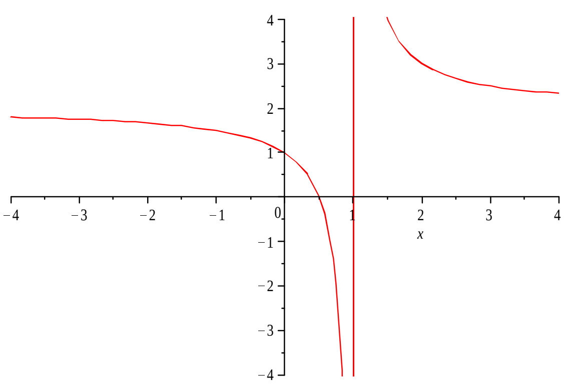
Add two to each of the ***y-values*** **of the previous function** $g(x)$**, as follows with** $f(x) \longmapsto g(x) \longmapsto h(x)$**,** $(-1,-1) \longmapsto (0,\ -1) \longmapsto (0,\ 1)$ **and** $(1,\ 1) \longmapsto (2,\ 1) \longmapsto (2,\ 3)$ **and the Horizontal Asymptote** $y = 0 \longmapsto y = 2$**.**

The Vertical Asymptote is left unchanged at $x = 1$ **and from left to right we are done.**

$f(x) = \frac{1}{x} \to f(x-1) = \frac{1}{x-1} \to f(x-1)+2 = \frac{1}{x-1}+2$, **as shown in order below.**







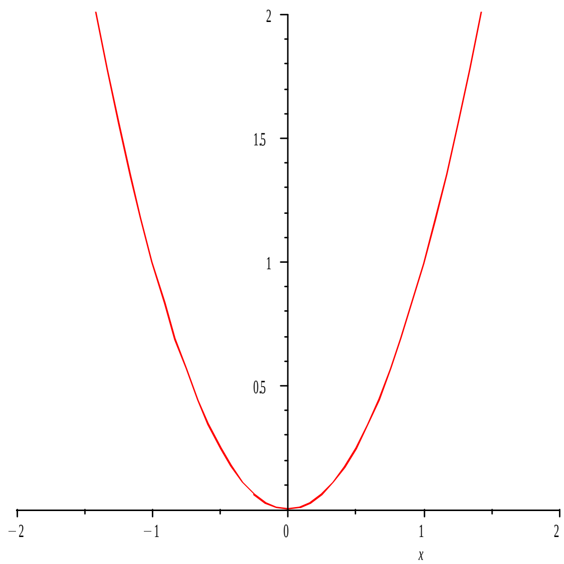
Reflections:

Reflection about the *y-axis*: Suppose that $f(x)$ is a function. If we multiply the inside of the function by -1, then the original function will be reflected about the *y-axis*. That is $f(-x)$ is a reflection of $f(x)$ about the *y-axis*.

Reflection about the *x-axis*: Suppose that $f(x)$ is a function. If we multiply the outside of the function by -1, then the original function will be reflected about the *x-axis*. That is $-f(x)$ is a reflection of $f(x)$ about the *x-axis*.

In the effort to maintain a minimalist approach, is it really necessary to memorize both reflections? No, not at all. Remember the first reflection stated is about the *y-axis,* and is a manipulation of the independent variable inside of the function. First, it is always wise to start with the independent variable, since every thing else depends on it. Recall the graph of the function $f(x) = x^2$. Notice that it is symmetric about the *y-axis*. Also notice that $f(-x) = (-x)^2 = x^2 = f(x)$. Multiplying x, the independent variable, on the inside of the function by -1 does not change the graph of the function. Memorize this fact and you have memorized the reflection about the *y-axis* and what happens when you multiply the inside of the function by -1. There is only one other axis, the *x-axis*. And there is only one other place that can be made negative, the outside of the function, otherwise known as y or the dependent variable. Memorize the properties of one well and you get the second one for free.

$y = f(x) = x^2$



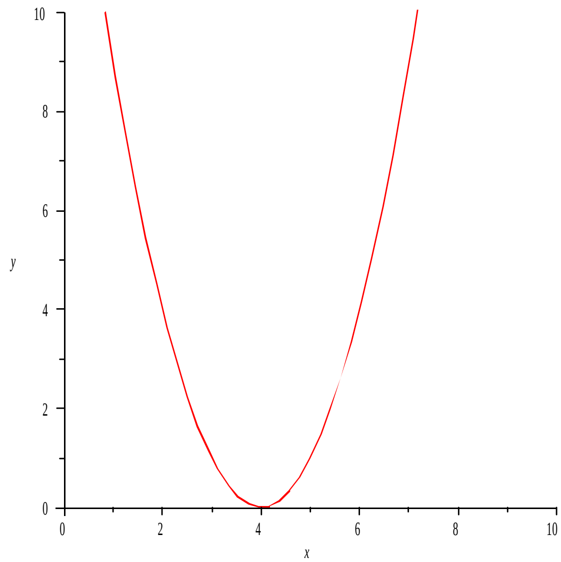
Example 2: Graph the function $h(x) = (4 - x)^2$

Recall we must start with the basic graph from which $h(x)$ **is composed. Here us** $f(x) = x^2$**. The horizontal translation calls for** $x - c$ **or** $x + c$ **, not** $c - x$ **or** $c + x$ **. Here order is important.**

Let $f(-x) = (-x)^2$**, which is a reflection about the *y-axis*.**

Finally, let $f(-(x-4)) = (-(x-4))^2 = (4-x)^2$**, which is a horizontal translation to the right four units. We must take stock in this as it could be confused with a horizontal shift to the left four units. Remember, we work from the inside out when doing Transformations.**

$h(x) = (4 - x)^2$



Example 3: Graph the function $h(x) = \sqrt{2 - x} + 1$

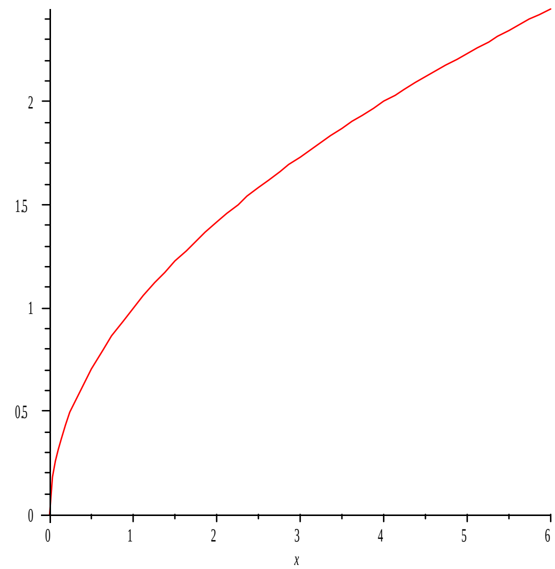
By this time we should have re-written the two previous examples several times. This one we do no different, but only to reinforce what we have already learned. In due time, all similar types of problems will begin to look alike.

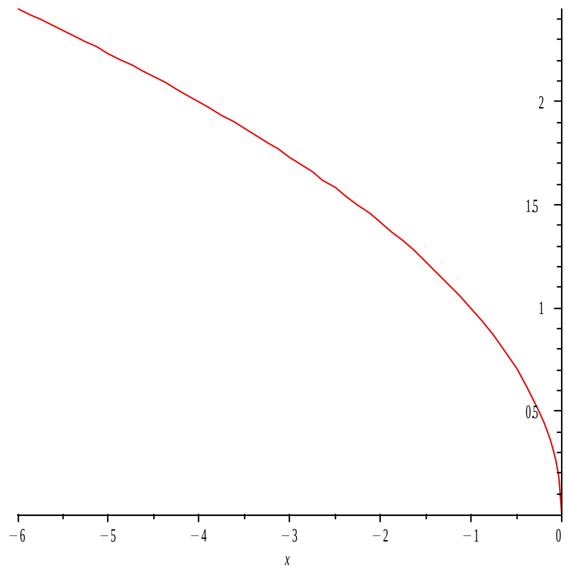
Let $f(x) = \sqrt{x}$. Working from the inside out, we have $g(x) = f(-x) = \sqrt{-x}$. This action reflects the function $f(x) = \sqrt{x}$ about the *y-axis.* Just as in the last example, we must remember that the horizontal shifts begin with x and add or subtract some constant, that is $x + c$ or $x - c$.

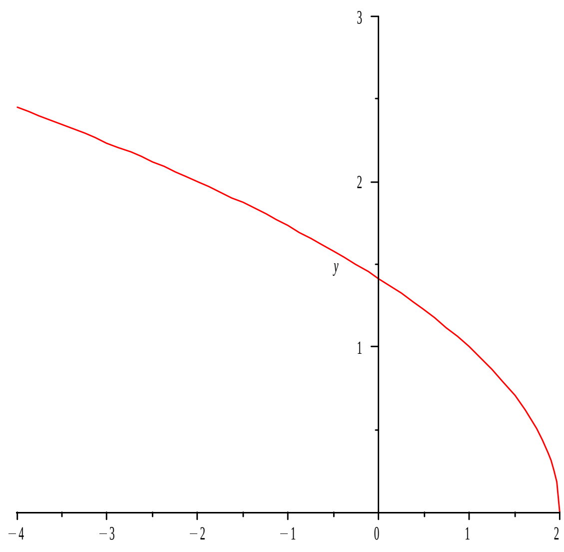
The next step in the evolution of the transformation is the horizontal shift $g(x - 2) = f(-(x - 2)) = \sqrt{-(x - 2)} = \sqrt{(2 - x)}$, a clear horizontal shift to the right by two units. It is completely understandable and normal to be confused by this step. The one and only remedy for this is practice. There is nothing wrong with practicing this same problem several times. Practice aloud. Practice in groups and demonstrate the problem to your classmates.

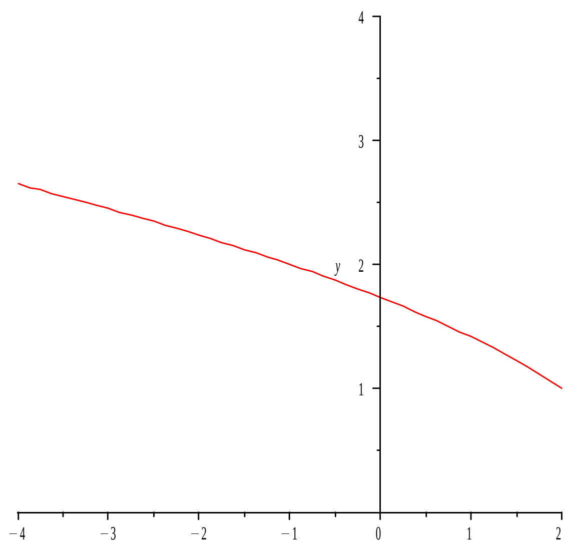
The last Transformation is to add 1 to the previous function, which will result in a vertical shift up 1 unit, and we are done.

$$h(x) = g(x-2)+1 = f(-(x-2))+1 = \sqrt{-(x-2)}+1 = \sqrt{(2-x)}+1.$$









Multiplication by a Constant: This is often referred to as Stretching and Compressing.

Multiplication of the Independent Variable by a Constant: Suppose that$c > 0$ **is a real number and that** $f(x)$ **is a function. The new function** $g(x) = f(c\cdot x)$ **is just** $f(x)$ **with** x **replaced by** $c\cdot x$**. Sometimes its easier to think of this action as multiplying each of the** ***x-coordinates*** **on the graph by** $\frac{1}{c}$.

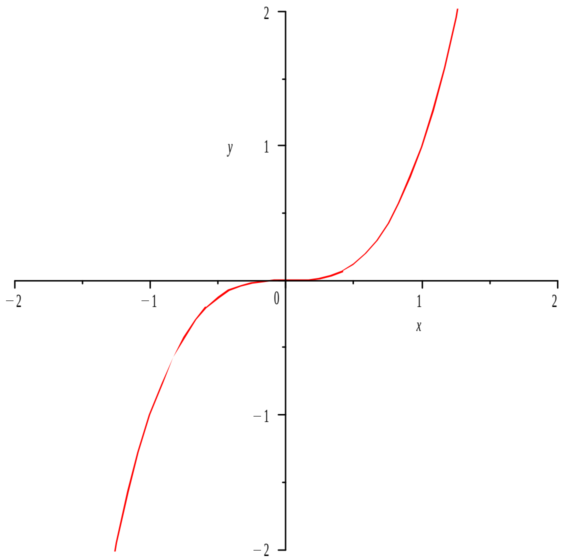
Multiplication of the Dependent Variable by a Constant: Suppose that$c > 0$ **is a real number and that** $f(x)$ **is a function. The new function** $g(x) = c\cdot f(x)$ **is just** $f(x)$ **multiplied by** c**. Sometimes its easier to think of this action as multiplying each of the** ***y-coordinates*** **on the graph by** c**.**

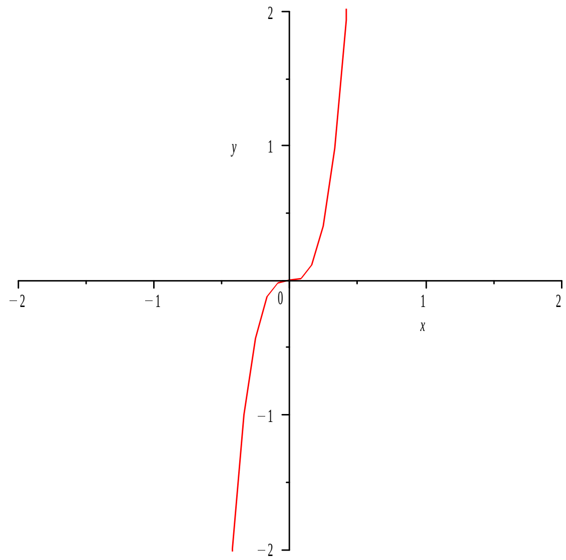
These are very simple Transformations, but often become convoluted when they are over contemplated. This happens when we start to think of them as Stretching and Compressing. This is not advised. The actions are very simple. Memorize these as they are stated above and study the given examples. There is no need to make them more difficult then they already are.

Example 4: Graph $h(x) = (3\cdot x)^3$

Begin with the basic function $f(x) = x^3 \longrightarrow h(x) = f(3\cdot x) = (3\cdot x)^3$**.**

Recall that the function $f(x) = x^3$ **has three fundamental points that we memorized in Chapter 1.1. In particular,** $f(x) = x^3$ **contains the points** $(-1, -1),\ (0,\ 0),\ (1,\ 1)$**. As with any Transformation we can simply adjust these points accordingly. For** $h(x) = (3\cdot x)^3$ **these points become,** $(-\frac{1}{3}, -1),\ (0,\ 0),\ (\frac{1}{3},\ 1)$ **, respectively. The evolution is shown in order here.**

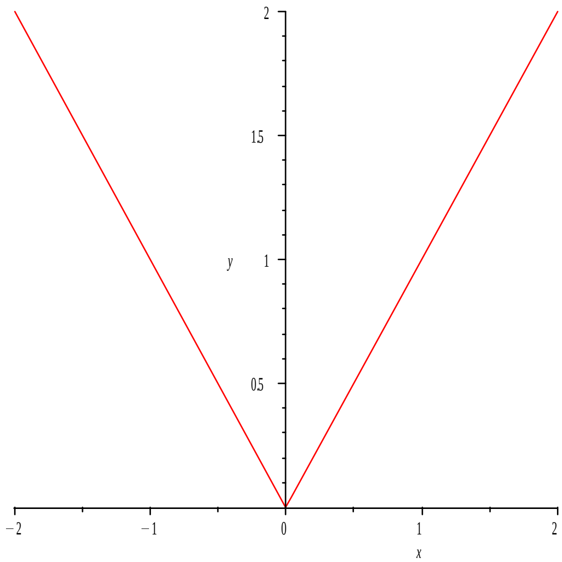


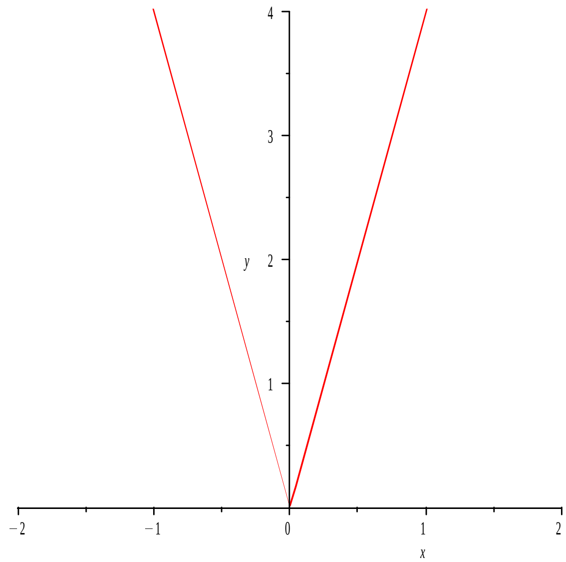


Example 5: Graph $h(x) = 4 \cdot |x|$

Again we start with the most obvious of the Seven Basic Graphs. That is, $f(x) = |x|$**, and we apply the relevant Transformation(s). Here** $h(x) = 4 \cdot f(x) = 4 \cdot |x|$**. This is about the simplest of all Transformations, as we need only multiply the dependent variable (or *y-coordinate*) by** 4**.**

Consider again the three fundamental points for this graph $(-1, 1),\ (0, 0),\ (1, 1)$**, and multiply each one of these *y-coordinates* by** 4**, and that is all. The result is** $(-1,\ 4),\ (0,\ 0),\ (1,\ 4)$**, respectively.**





Notice that this is also a Compression of the original graph. That is because $f(4\cdot x) = 4\cdot f(x) = |4\cdot x| = 4\cdot |x|$. This is exactly the kind of confusion we are trying to avoid in this lesson. *Some time this works, or some times that works when this or that is true.* In mathematics it makes much more sense to have statements that work all of the time, as opposed to some that work more easily some times, but then need to be adjusted depending on the situation. Mathematics is a discipline of absolutes. When two is added to two, the sum is four. All things mathematical should be treated as such.

Example 6: Graph $h(x) = 3x^2 - 6x + 4$ by first Completing the Square and writing the function in Vertex Form and then using Transformations.

First we Complete the Square to find Vertex Form.

$3\cdot(x^2 - 2x) + 4$

$= 3\cdot(x^2 - 2x + 1) + 4 - 3$

$= 3\cdot(x - 1)^2 + 1$

Thus $h(x) = 3 \cdot (x-1)^2 + 1$

We approach this no different than the last five examples. We know that we will be starting with the basic function $f(x) = x^2$**with the points** $(-1,\ 1),\ (0,\ 0),\ (1,\ 1)$ **and we take it one step at a time, from the inside we work out.**

First we perform a Horizontal Transformation to the right by a unit of one $f(x-1) = (x-1)^2$**, so our respective points become** $(0, 1),\ (1, 0),\ (2, 1)$**.**

Second, we multiply all of the the previous *y-coordinates* by three, $3 \cdot f(x-1) = 3 \cdot (x-1)^2$**, so our respective points become** $(0, 3),\ (1, 0),\ (2, 3)$**.**

Finally, we add one to the last function to yield a Vertical Transformation up one unit, $3 \cdot f(x-1)+1 = 3 \cdot (x-1)^2+1$**. Our respective points become** $(0,\ 4),\ (1,\ 1),\ (2,\ 4)$**.**

This gives us our $h(x) = 3x^2 - 6x + 4$ **or** $h(x) = 3 \cdot (x-1)^2 + 1$ **and we are done.**

Notice that the vertex is $(h,\ k) = (1,\ 1)$**.**