1.2 Linear Equations and Inequalities

Math students often take fundamental concepts for granted. This is one of the most common mistakes that inhibits mathematical development. Students may demonstrate an ability to perform advanced techniques and at the same time lack the skills required to perform basic tasks. The result is a superficial knowledge of the game and an inability to communicate concepts and complete solutions correctly. There is no sense in doing calculus if the solution collapses in the end due to a faulty understanding of basic principles. This deficiency is similar to playing a beautiful piece of music that is littered with instances of cacophony, or being a boxer with a highly refined left-hook who lacks the ability to jab, block, and maneuver. These analogies all have a common thread and that is the need to develop a second-nature understanding of basic principles in order to always perform well. If you want to succeed in calculus, you must acquire a conditional response when confronting steps that depend upon the application of fundamental concepts. To attain this, we must practice the fundamentals on a regular basis. Remember that a math problem is done well if necessary and sufficient conditions are met. Students often focus too much on the solution and not enough on the fundamental steps taken to produce a solution. The point of a sentence is not the final punctuation, but rather the substance within the sentence. A mathematical solution is only as good as the quality of work required to solve the problem.

In this section we will discuss the properties of lines. Here is a profoundly basic concept that pre-calculus students tend to observe yet not admire. A closer examination is required so that we may dissect the concept and take only the necessary pieces. Once we have those pieces, we drill on them with repetition and rigor until they become second nature (conditional response). Repetition of fundamental concepts is the key to understanding mathematics, and from it's daily discipline we will achieve great success. Writing and re-writing is the most efficient approach to understanding mathematics. This method does take time, so be patient with yourself and remember that there is a difference between what is difficult and what is time consuming.

The linear equation has two basic forms and each serve a separate purpose.

1. The Slope-Intercept Form: $y = m \cdot x + b$

Here m **represents the slope and** b **is the** ***y-intercept.***

2. The Point-Slope Form: $y - y_1 = m \cdot (x - x_1)$ **or** $y = m \cdot (x - x_1) + y_1$

Here m **is still slope and** $(x_1,\, y_1)$ **are the given points.**

We always want to begin with point-slope form and simplify to obtain slope-intercept form. Beginning with $y = m \cdot x + b$ **nearly always results in an incorrect solution. It has little functional purpose other than being a simplified linear form. Familiarize yourself with the point-slope form so that when asked for the equation of a line, invoking it is the conditioned response.**

Example 1: Find the equation for the line with a slope of 4 **and passing through the point** $(2,\, 3)$**. Put the final result in the slope-intercept form.**

Recall that we always want to begin with the point-slope form. We have all of the values we need. The problem only requires that we put these values in the correct spot. When substituting values into an expression, we must get into the habit of replacing the values with blank parentheses first. It may seem pedantic, but this practice will yield great results over time.

$$y - y_1 = m \cdot (x - x_1) \implies y - (\ ) = (\ ) \cdot (x - (\ )) \implies y - 3 = 4 \cdot (x - 2)$$

Here it is tempting to simplify without giving deliberate thought to the process. This is a big mistake. It is important that we refine our practice by taking a step back and sounding out each step.

First distribute the 4**, so that** $y - 3 = 4x - 8$**.**

Now MOVE the -3 **from the left-hand side of the equation to the right; by doing so it becomes positive** 3**.**

Then $y = 4x - 8 + 3 \implies y = 4x - 5$, and we are done.

Re-write this problem at least three times, or as long as it takes for you to write it perfectly on your own. It is safe to replace the given values with other numerals, but it is not necessary. The mechanics of each step is what requires the most attention here. It does not matter what the values are. Remember you are just moving objects around a board according to the two operations of arithmetic as discussed in the previous section.

Example 2: Find the equation of the line passing through the points $(1,\ 10)$, and $(-2,\ 1)$.

Note that whenever we are asked to give the equation of a line, we will assume that the final result should be simplified to slope-intercept form.

The example asks for the equation of a line, so we immediately write down the general point-slope form as a result of proper training and conditional response.

$$y - y_1 = m \cdot (x - x_1)$$

For the points $(x_1,\ y_1)$ we may choose which ever points we want from the two given. Since both points are on the line, both will work. However, we need to find the slope. This is done as follows.

The slope is equal to the change in y over the change in x, or rise over run. This is also written as *delta-y* ($\triangle y$) over *delta-x* ($\triangle x$).

Given the points on a line $(x_1,\ y_1)$, and $(x_2,\ y_2)$, the slope is as follows.

$$m = \frac{rise}{run} = \frac{\triangle y}{\triangle x} = \frac{y_1 - y_2}{x_1 - x_2} = \frac{y_2 - y_1}{x_2 - x_1}$$

It is very important to notice that when we start with y_1 in the numerator then we must start with x_1 in the denominator. Likewise, if we started with y_2 in the numerator, then we must start with x_2 in the denominator. The order of the points must remain consistent.

$m = \dfrac{(10) - (1)}{(1) - (-2)} = \dfrac{9}{3} = 3$ **and** $m = \dfrac{(1) - (10)}{(-2) - (1)} = \dfrac{-9}{-3} = 3$.

Recall that we always use parentheses when substituting values into an expression. Here we can see one important reason why we do this. The value $x_2 = -2$ could become problematic when subtracting a negative value. In fact there is a plethora of reasons to use parentheses here, but mostly it is a measure we take in order to avoid making small, yet regretful, mistakes. It also alters our perspective and will become invaluable in the future.

Using $(x_1,\, y_1) = (1,\, 10)$ we have

$$y - y_1 = m \cdot (x - x_1) \implies y - (\ ) = (\ ) \cdot (x - (\ )) \implies y - 10 = 3 \cdot (x - 1).$$

It is important that we perform similar problems the same way every time in order to reinforce a conditional response. We then to continue in the same way as we finished Example 1.

First distribute the 3 on the right, so that $y - 10 = 3x - 3$.

Now MOVE the -10 from the left-hand side of the equation to the right; by doing so it becomes positive 10.

Then $y = 3x - 3 + 10 \implies y = 3x + 7$ and we are done.

Re-write this problem at least three times, or as long as it takes for you to write it perfectly on your own. It is safe to replace the given values with other numerals, but it is not necessary. The mechanics of each step is what requires the most attention here. It does not matter what the values are. Remember you are just moving objects around a board according to the two operations of arithmetic as discussed in the previous section.

Mantras: Facts worth repeating out loud regarding intercepts.

The *x-intercept* occurs when $y = 0$.

The *y-intercept* occurs when $x = 0$.

Throughout this book we will expose certain mantras, which are simply facts that are worthy of repeating. You want to repeat the fact that the ***x-intercept*** occurs when $y = 0$. You repeat these mantras whenever you think of it or whenever the topic comes up. This will reinforce the conditional response. It will be without question during an exam where the ***x-intercept*** or the ***y-intercept*** occur. Note that we always start with x. This is important, since the ***y-value*** depends on the ***x-value***.

Given a linear equation, which is exactly when the highest power of x is one, we have the following:

If $y = mx + b$, then the ***x-intercept*** occurs when $y = 0$.

Then $0 = mx + b \implies -b = mx \implies \dfrac{-b}{m} = x$.

Now we have a general form for the ***x-intercept*** of a linear function given in slope-intercept form. Take caution when using these general forms because they distract from the practice that we so desperately need.

Also, if $y = mx + b$, then the ***y-intercept*** occurs when $x = 0$.

Then $y = m \cdot 0 + b \implies y = b$.

Mantras: Facts worth repeating regarding slope relationships of lines.

Parallel lines have the same slope.

Perpendicular line have negative reciprocal slope.

This last remark concerning perpendicular lines is a little ambiguous. If $y = m_1x + b_1$ and $y = m_2x + b_2$ are perpendicular lines, then their slopes are such that $m_1 = -\dfrac{1}{m_2}$ and $m_2 = -\dfrac{1}{m_1}$.

Solving Linear Equations

Solving linear equations for x is simply an exercise in the laws of basic arithmetic. These examples should not be overlooked. This is the bare minimum of what you need to know - the essentials.

Example 3: Solve the following equation for x.

$3(4x-2)=4(x+1)$

Take it slow and sound out each step.

Distribute the 3 on the left-hand side and the 4 on the right-hand side.

$12x-6=4x+4$

Now move the $4x$ from right to left and -6 from left to right, as follows.

$12x-4x=4+6$

Combine like terms on each side.

$8x=10$

Move the 8 from the left to the right.

$$x=\frac{10}{8}$$

Reduce the fraction to lowest terms.

$$x=\frac{2\cdot 5}{2\cdot 4} \implies x=\frac{5}{4}$$

It may seem pedantic to sound out each step, but if we re-write the above verbatim several times then we can do any problem similar to it because we memorized the mechanics of the problem. It cannot be stressed enough how essential this is to your success.

Example 4: Solve the following equation for x.

$2(3x-7)=5-3(2x+4)$

Distribute the 2 on the left and the -3 on the right.

Failing to distribute to all of the terms correctly is one of the most common simple mistakes made in all levels of mathematics. Always keep this in mind when distributing.

$6x - 14 = 5 - 6x - 12$

Combine like terms on the right.

$6x - 14 = -7 - 6x$

Move -14 from the left to he right and $-6x$ from the right to the left, as follows.

$6x + 6x = -7 + 14$

Combine like terms on both sides.

$12x = 7$

Move 12 from the left to the right.

$x = \frac{7}{12}$

Important Note:

If your final result is always true, like $10 = 10$, then the solution for x is all real numbers.

If your final result is never true, like $8 = 7$, then there is no solution for x.

Equations and inequality operations only differ in one respect. With an inequality, if you multiply (or divide) by a negative number you must flip (or reverse) the inequality operator. The solution for a linear equation is nearly always a point, except if the solution is all real numbers or no solutions. An inequality will nearly always be an interval, except when the solution is all real numbers or no solutions. Therefore, inequalities are treated just like equations except when multiplying (or dividing) by a negative number or when writing a solution.

Recall that $x > a$ reads as x is strictly greater than a, so that it will never take on a as a value but will come infinitely close.

This is expressed in interval notation as (a, ∞). The open parenthesis signifies non-inclusion of a in the solution set.

Always use an open parenthesis with positive or negative infinity.

Remember that you cannot include positive or negative infinity because infinity is a concept and not a number.

The solution is expressed graphically on the number line below. The open circle represents non-inclusion of a**.**

$x < b$ **reads as** x **is strictly less than** b**, so that it will never take on** b **as a value but will come infinitely close.**

This is expressed in interval notation as $(-\infty, b)$**. The open parenthesis signifies non-inclusion of** b **in the solution set.**

Always use an open parenthesis with positive or negative infinity. Remember that you can not include positive or negative infinity because infinity is a concept and not a number.

The solution is expressed graphically on the number line below. The open circle represents non-inclusion of b**.**

$x \geq c$ **reads as** x **is greater than or equal to** c**, so that it will take on** c **as a value.**

This is expressed in interval notation as $[c, \infty)$**. The closed bracket signifies the inclusion of** c **in the solution set.**

Always use an open parenthesis with positive or negative infinity. Remember that you cannot include positive or negative infinity because infinity is a concept and not a number.

The solution is expressed graphically on the number line below. The closed circle signifies inclusion of c.

$x \leq d$ **reads as** x **is less than or equal to** d**, so that it will take on** d **as a value.**

This is expressed in interval notation as $(-\infty,\, d]$**. The closed bracket signifies the inclusion of** d **in the solution set.**

Always use an open parenthesis with positive or negative infinity. Remember that you can not include positive or negative infinity because infinity is a concept and not a number.

The solution is expressed graphically on the number line below. The closed circle signifies inclusion of d.

Example 5: Solve the following inequality for x.

$2x + 3 > 4x - 7$

Move 3 **from the left to the right and move** $4x$ **from the right to the left.**

$2x - 4x > -7 - 3$

Combine like terms on the left and right.

$-2x > -10$

Move the -2 **from the left side to the right. In doing so we must flip the inequality.**

$x < \dfrac{-10}{-2} \implies x < 5$

The solution is given in interval notation as $(-\infty, 5)$**.**

The above is expressed graphically on the number line below.

Example 6: Solve the following inequality for x**.**

$3(x + 2) - 5 \geq 7$

Move the -5 **from left to right.**

$3(x + 2) \geq 7 + 5$

Distribute the 3 **on the left and combine like terms on the right.**

$3x + 6 \geq 12$

Move the 6 **from the left to the right.**

$3x \geq 12 - 6$

Combine like terms on the right.

$3x \geq 6$

Move the 3 **from the left to the right.**

$x \geq \dfrac{6}{3} \implies x \geq 2$

The solution is given in interval notation as $(2, \infty)$.

The above is expressed graphically on the number line below.